IJESM
Consulting, help, relaxation
INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES
&
MANAGEMENT

MAPPINGS OF FUZZY METRIC SPACE AND PROVED THE COMMON FIXED POINT THEOREM

C.S.Chauhan
Department Of Applied Science (Appl. Mathematics), Institute of Engineering and Technology
DAVV Indore (M.P.), India

ABSTRACT

Many authors modified Fuzzy metric space and proved fixed point results in Fuzzy metric space. Singh B. and Chauhan were first introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. Cho et al were introduced the concept of compatible mapping of type (P). In this paper, a fixed point theorem for six self-mappings is presented by using the concept of compatible maps of type(P) which is the generalized result using common fixed point theorem.

Keywords: Common fixed points, fuzzy metric space, compatible maps, and weak compatible maps..

INTRODUCTION

The concept of Fuzzy sets was initially investigated as a new way to represent vagueness in everyday life. Subsequently, it was developed by many authors and used in various fields. To use this concept in Topology and Analysis, several researchers have defined Fuzzy metric space in various ways. In recently, proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. The concept of compatible mapping of type (P) proved a fixed point theorem for six self maps in a fuzzy metric space [1-2] and [5-7]. In this paper, a fixed point theorem for six self maps has been established using the concept of compatible maps of type (P), which generalizes the result. For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

PRELIMINARIES

In preliminaries, we analysis some fundamentals of basic fuzzy metric spaces, which be used the rest of our paper.

Definition 2.1. A binary operation $*$: $[0,1]\times[0,1]\rightarrow[0,1]$ is called a *t-norm* if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for a, b, c, d $\in$ [0, 1].

Definition 2.2. The 3-tuple (X, M,$*$) is said to be a *Fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a Fuzzy set in [ ) 2 $X \times 0,\infty$ satisfying the following conditions: for all x, y, z $\in$ X and s, t > 0

(FM-1) M (x, y, 0) = 0,
(FM-2) M (x, y, t) =1 for all t > 0 if and only if x = y,
(FM-3) M (x, y, t) = M (y, x, t),
(FM-4) M (x, y, t) $*$ M (y, z, s) $\le$ M (x, z, t + s),
(FM-5) M (x, y, .) : $[0,\infty) \rightarrow [0, 1]$ is left continuous,
(FM-6) $\lim_{t\to\infty}$ M (x, y, t) =1.

Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1 for all t > 0. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [9] Let (X, d) be a metric space. Define a $*$ b = min {a, b} and

$$M(x,y,t) = \frac{t}{t + d(x,y)}$$

for all x,y $\in$t and t > 0. Then (X, M,$*$) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3. [9] A sequence{ } *n x* in a Fuzzy metric space (X, M,$*$) is said to be a *Cauchy sequence* if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n0 \in N$ such that
M (xn, xm , t) > 1 - ε for all n, m $\geq$ n0.
The sequence {xn} is said to *converge* to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exists n0$\in$N such that M (xn, x, t) > 1 - ε for all n,m $\geq$ n0.
A Fuzzy metric space (X, M,$*$) is said to be *complete* if every Cauchy sequence in it converges to a point in it.

Definition 2.4. [11] Self mappings A and S of a Fuzzy metric space (X, M,$*$) are said to be *compatible* if and only if M (ASxn, SAxn, t)$\to$1 for all t > 0, whenever {xn} is a sequence in X such that Sxn, Axn $\to$p for some p in X as n $\to\infty$.

Definition 2.5. [10] Self maps A and S of a Fuzzy metric space (X, M,$*$) are said to be compatible of type (P) M (AAxn, SSxn, t) $\to$1 for all t >0, whenever {xn} is a sequence in X such that Sxn, Axn $\to$p for some p in X as n $\to\infty$.

Proposition 2.1. [5] In a fuzzy metric space (X, M, $*$) limit of a sequence is unique.

Lemma 2.1. [4] Let (X, M,$*$) be a fuzzy metric space. Then for all x, y$\in$X, M (x, y,.)
Lemma 2.2. [1] Let (X, M,$*$) be a fuzzy metric space. If there exists k$\in$(0, 1) such that for all x, y$\in$X, M (x, y, kt) $\geq$M (x, y, t) $\forall$ t > 0, then x = y.

Lemma 2.3. [5] Let {xn} be a sequence in a fuzzy metric space (X, M,$*$). If there exists a number k$\in$(0, 1) such that M (xn+2, xn+1, kt) $\geq$M (xn+1, xn, t) $\forall$ t > 0 and n$\in$N. Then {xn} is a Cauchy sequence in X.

Lemma 2.4. [7] The only t-norm$*$ satisfying $r * r \geq r$ for all r$\in$[0, 1] is the minimum t-norm, that is $a * b = \min\{a, b\}$ for all a, b$\in$[0, 1].

Lemma 2.5. [3] Let (X, M,$*$) be a fuzzy metric space if there exists q$\in$(0,1) such that M (x, y, qt) $\geq$ M (x, y, t/qn) for integer n. Taking limit as n $\to\infty$, M (x, y, t) $\geq$ 1 and hence x = y.

MAIN RESULTS

Theorem 3.1. Let (X, M, $*$) be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied:
(a) P(X) $\subset$ST(X), Q(X) $\subset$AB(X);
(b) AB = BA, ST = TS, PB = BP, QT = TQ;
(c) Either AB or P is continuous;
(d) Pair (P, AB) is compatible and (Q, ST) is compatible map of type (P);
(e) There exists q$\in$(0, 1) such that for every x, y$\in$X and t > 0
M (Px, Qy, qt) $\geq$ M (ABx, STy, t) $*$ M (Px, ABx, t) $*$ M (Qy, STy,t) $*$ M (Px, STy, t). Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof.
Let x0$\in$X. From (a) there exist x1, x2$\in$X such that Px0 = STx1 and Qx1 = ABx2.
Inductively, we can construct sequences {xn} and {yn} in X such that:
Px2n-2 = STx2n-1 = y2n-1 and Qx2n-1 = ABx2n = y2n for n = 1, 2, 3...

Step 1. Put x = x2n and y = x2n+1 in (e), we get
M (Px2n, Qx2n+1, qt) $\geq$ M (ABx2n, STx2n+1, t) $*$ M (Px2n, ABx2n, t) $*$ M (Qx2n+1,STx2n+1,t) $*$ M(Px2n, STx2n+1, t).
= M (y2n, y2n+1, t) $*$M (y2n+1, y2n, t) $*$M(y2n+2, y2n+1, t) $*$M (y2n+1, y2n+1, t)
$\geq$ M (y2n, y2n+1, t) $*$ M (y2n+1, y2n+2, t).
From lemmas 2.1 and 2.2, we have
M (y2n+1, y2n+2, qt) $\geq$ M (y2n, y2n+1, t).
Similarly, we have
M (y2n+2, y2n+3, qt) $\geq$ M (y2n+1, y2n+2, t).
Thus, we have
M (yn+1, yn+2, qt) $\geq$ M (yn, yn+1, t) for n = 1, 2,...
M (yn, yn+1, t) $\geq$ M (yn, yn+1, t/q)

$\geq$ M (yn-2, yn-1, t/q2)
$\geq$ M (y1, y2, t/qn)$\rightarrow$1 as n $\rightarrow \infty$,and
hence M (yn, yn+1, t) $\rightarrow$1 as n $\rightarrow \infty$ for any t > 0.
For each ε > 0 and t > 0, we can choose n0$\in$N such that
M (yn, yn+1, t) > 1 - ε for all n > n0
For m, n$\in$N, we suppose m$\geq$ n. Then we have
M (yn, ym, t) $\geq$ M (yn, yn+1, t/m-n)$*$ M (yn+1, yn+2, t/m-n) $*$ ... $*$ M (ym-1, ym, t/m-n)
$\geq$ (1 - ε) $*$ (1 - ε) $*$ ... $*$ (1 - ε) (m - n) times
$\geq$ (1 - ε) and hence {yn} is a Cauchy sequence in X.
Since (X, M, $*$) is complete, {yn} converges to some point z $\in$ X. Also its subsequences converge to the same point z$\in$X.
i.e. {Qx2n+1} $\rightarrow$z and{STx2n+1} $\rightarrow$z (1)
{Px2n} $\rightarrow$z and {ABx2n } $\rightarrow$z. (2)

Case I. Suppose AB is continuous.
Since AB is continuous, we have (AB)2x2n $\rightarrow$ABz and ABPx2n $\rightarrow$ABz.
As (P, AB) is compatible pair of type (P), we have
M (PPx2n, ABABx2n,t) =1 for all t > 0 or M (PPx2n,ABz,t) =1.
Therefore PPx2n $\rightarrow$ABz

Step 2. Put x = ABx2n and y = x2n+1 in (e), we get
M (PABx2n, Qx2n+1, qt) $\geq$ M (ABABx2n, STx2n+1, t) $*$ M (PABx2n, ABABx2n, t)
$*$ M (Qx2n+1, STx2n+1, t) $*$ M (PABx2n, STx2n+1, t).
Taking n $\rightarrow \infty$, we get
M (ABz, z, qt) $\geq$ M (ABz, z, t) $*$ M (ABz, ABz, t) $*$ M (z, z, t) $*$ M (ABz, z, t)
$\geq$ M (ABz, z, t) $*$ M (ABz, z, t) i.e. M(ABz, z, qt) $\geq$M(ABz, z, t).
Therefore, by using lemma 2.2, we get ABz = z……… (3)

Step 3. Put x = z and y = x2n+1 in (e), we have
M (Pz, Qx2n+1, qt) $\geq$ M (ABz, STx2n+1, t) $*$ M (Pz, ABz, t)
$*$ M (Qx2n+1, STx2n+1, t) $*$ M (Pz, STx2n+1, t).
Taking n $\rightarrow \infty$ and using equation (1), we get
M (Pz, z, qt) $\geq$ M (z, z, t) $*$ M (Pz, z, t) $*$ M (z, z, t) * M (Pz, z, t)
$\geq$ M (Pz, z, t) $*$ M (Pz, z, t) i.e. M(Pz, z, qt) $\geq$ M(Pz, z, t).
Therefore, by using lemma 2.2, we get Pz = z. Therefore, ABz = Pz = z.

Step 4. Putting x = Bz and y = x2n+1 in condition (e), we get
M (PBz, Qx2n+1, qt) $\geq$ M (ABBz, STx2n+1, t) $*$ M (PBz, ABBz, t)
$*$ M (Qx2n+1, STx2n+1, t) $*$ M (PBz,STx2n+1, t).
As BP = PB, AB = BA, so we have P (Bz) = B (Pz) = Bz and (AB)(Bz) = (BA)(Bz) = B (ABz) = Bz.
Taking n $\rightarrow \infty$ and using (1), we get
M (Bz, z, qt) $\geq$ M (Bz, z, t) $*$ M (Bz, Bz, t) $*$ M (z, z, t) $*$ M (Bz, z, t)
$\geq$ M (Bz, z, t) $*$ M (Bz, z, t) i.e. M(Bz, z, qt) $\geq$M(Bz, z, t).
Therefore, by using lemma 2.2, we get
Bz = z and also we have ABz = z . Az = z.
Therefore, Az = Bz = Pz = z……………….. (4)

Step 5. As P (X) $\subset$ST (X), there exists u$\in$X such that z = Pz = STu.
Putting x = x2n and y = u in (e), we get
M (Px2n, Qu, qt) $\geq$ M (ABx2n, STu, t) $*$ M (Px2n, ABx2n, t)
$*$ M (Qu, STu, t) $*$ M (Px2n, STu, t).
Taking n $\rightarrow \infty$ and using (1) and (2), we get
M (z, Qu, qt) $\geq$ M (z, z, t) $*$ M (z, z, t) $*$ M (Qu, z, t) $*$ M (z, z, t)
$\geq$ M (Qu, z, t) i.e. M(z, Qu, qt) $\geq$ M(z, Qu, t).
Therefore, by using lemma 2.2, we get Qu = z. Hence STu = z = Qu. Since (Q, ST) is compatible pair of type (P), therefore, by proposition 2.2, we have QSTu = STQu.
Thus Qz = STz.

Step 6. Putting $x = x_{2n}$ and $y = z$ in (e), we get
$M(Px_{2n}, Qz, qt) \geq M(ABx_{2n}, STz, t) * M(Px_{2n}, ABx_{2n}, t)$
$* M(Qz, STz, t) * M(Px_{2n}, STz, t)$.
Taking $n \to \infty$ and using (2) and step 5, we get
$M(z, Qz, qt) \geq M(z, Qz, t) * M(z, z, t) * M(Qz, Qz, t) * M(z, Qz, t)$
$\geq M(z, Qz, t) * M(z, Qz, t)$ i.e. $M(z, Qz, qt) = M(z, Qz, t)$.
Therefore, by using lemma 2.2, we get $Qz = z$.

Step 7. Putting $x = x_{2n}$ and $y = Tz$ in (e), we get
$M(Px_{2n}, QTz, qt) \geq M(ABx_{2n}, STTz, t) * M(Px_{2n}, ABx_{2n}, t)$
$* M(QTz, STTz, t) * M(Px_{2n}, STTz, t)$.
As $QT = TQ$ and $ST = TS$, we have $QTz = TQz = Tz$ and $ST(Tz) = T(STz) = TQz = Tz$.
Taking $n \to \infty$, we get
$M(z, Tz, qt) \geq M(z, Tz, t) * M(z, z, t) * M(Tz, Tz, t) * M(z, Tz, t)$
$\geq M(z, Tz, t) * M(z, Tz, t)$ i.e. $M(z, Tz, qt) = M(z, Tz, t)$.
Therefore, by using lemma 2.2, we get $Tz = z$. Now $STz = Tz = z$ implies $Sz = z$.
Hence $Sz = Tz = Qz = z$. (5)
Combining (4) and (5), we get $Az = Bz = Pz = Qz = Tz = Sz = z$. Hence, z is the common fixed point of A, B, S, T, P and Q.

Case II. Suppose P is continuous. As P is continuous, $P^2x_{2n} \to Pz$ and $P(AB)x_{2n} \to Pz$.
As (P, AB) is compatible pair of type (P), we have
$M(PPx_{2n}, ABABx_{2n}, t) = 1$ for all $t > 0$ or $M(Pz, ABABx_{2n}, t) = 1$
Therefore $(AB)Px_{2n} \to Pz$.

Step 8. Putting $x = Px_{2n}$ and $y = x_{2n+1}$ in condition (e), we have
$M(PPx_{2n}, Qx_{2n+1}, qt) \geq M(ABPx_{2n}, STx_{2n+1}, t) * M(PPx_{2n}, ABPx_{2n}, t)$
$* M(Qx_{2n+1}, STx_{2n+1}, t) * M(PPx_{2n}, STx_{2n+1}, t)$.
Taking $n \to \infty$, we get
$M(Pz, z, qt) \geq M(Pz, z, t) * M(Pz, Pz, t) * M(z, z, t) * M(Pz, z, t)$
$\geq M(Pz, z, t) * M(Pz, z, t)$ i.e. $M(Pz, z, qt) \geq M(Pz, z, t)$.
Therefore by using lemma 2.2, we have $Pz = z$. Further, using steps 5, 6, 7, we get $Qz = STz = Sz = Tz = z$.

Step 9. As $Q(X) \subset AB(X)$, there exists $w \in X$ such that $z = Qz = ABw$.
Put $x = w$ and $y = x_{2n+1}$ in (e), we have
$M(Pw, Qx_{2n+1}, qt) \geq M(ABw, STx_{2n+1}, t) * M(Pw, ABw, t)$
$* M(Qx_{2n+1}, STx_{2n+1}, t) * M(Pw, STx_{2n+1}, t)$.
Taking $n \to \infty$, we get
$M(Pw, z, qt) \geq M(z, z, t) * M(Pw, z, t) * M(z, z, t) * M(Pw, z, t)$
$\geq M(Pw, z, t) * M(Pw, z, t)$ i.e. $M(Pw, z, qt) \geq M(Pw, z, t)$.
Therefore, by using lemma 2.2, we get $Pw = z$.Therefore, $ABw = Pw = z$. As (P, AB) is compatible pair of type (P), we have $Pz = ABz$. Also, from step 4, we get $Bz = z$.
Thus, $Az = Bz = Pz = z$ and we see that z is the common fixed point of the six maps in this case also.

Uniqueness: Let u be another common fixed point of A, B, S, T, P and Q.
Then $Au = Bu = Pu = Qu = Su = Tu = u$. Put $x = z$ and $y = u$ in (e), we get
$M(Pz, Qu, qt) \geq M(ABz, STu, t) * M(Pz, ABz, t) * M(Qu, STu, t) * M(Pz, STu, t)$.
Taking $n \to \infty$, we get
$M(z, u, qt) \geq M(z, u, t) * M(z, z, t) * M(u, u, t) * M(z, u, t)$
$\geq M(z, u, t) * M(z, u, t)$ i.e. $M(z, u, qt) \geq M(z, u, t)$.
Therefore by using lemma 2.2, we get $z = u$. Therefore z is the unique common fixed point of self-maps A, B, S, T, P and Q.
If we take $B = T = I$, the identity map on X in Theorem 3.1, then
condition (b) is satisfied trivially and we get

Corollary 3.1. Let (X, M, $*$) be a complete fuzzy metric space and let A, S, P and Q be mappings from X into itself such that the following conditions are satisfied:
(a) $P(X) \subset S(X)$, $Q(X) \subset A(X)$;
(b) either A or P is continuous;
(c) (P, A) and (Q, S) is compatible maps of type (P);
(d) there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$
$M(Px, Qy, qt) \geq M(Ax, Sy, t) * M(Px, Ax, t) * M(Qy, Sy, t) * M(Px, Sy, t)$.
Then A, P, S and Q have a unique common fixed point in X

REFERENCES

1. Grabiec, M., Fixed points in Fuzzy metric space, Fuzzy sets and systems, 27(1998), 385-389.
2. Jain, A. and Singh, B., A fixed point theorem using weak compatible in fuzzy metric space,VarahmihirJ. Math. Sci.Vol.5 No. I(2005), 297-306.
3. Jungck, G., Murthy, P.P. and Cho, Y.J., Compatible mappings of type (A) and common fixedpoints,Math. Japonica, 38 (1993), 381-390.
4. Klement, E.P., Mesiar, R. and Pap, E., Triangular Norms, Kluwer Academic Publishers.
5. Kramosil, I. and Michalek, J., Fuzzy metric and statistical metric spaces,Kybernetica 11 (1975),336-344.
6. Cho, S.H., On common fixed point theorems in fuzzy metric spaces, J. Appl. Math. & ComputingVol. 20 (2006), No. 1 -2, 523-533.
7. Cho, Y.J., Fixed point in Fuzzy metric space, J. Fuzzy Math. 5(1997), 949-962.
8. George, A. and Veeramani, P., On some results in Fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994), 395-399.
9. Mishra, S.N., Mishra, N. and Singh, S.L., Common fixed point of maps in fuzzy metric space,Int.J. Math. Math. Sci. 17(1994), 253-258.
10. Pathak, Chang, Cho, Fixed point theorems for compatible map of type (P), Indian Journal ofMath., 36(2), (1994), 151-166.